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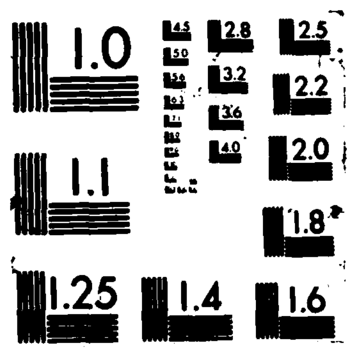
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Suppose that $(X_i,\tilde{Y}_i)$, $i = 1, \ldots, n$, are iid. samples of $(X,\tilde{Y})$. Instead of $\tilde{Y}_i$, we can only observe $Y_i = \max(\tilde{Y}_i,0)$. Denote by $m(x)$ the median-regression function of $\tilde{Y}$ with respect to X. This paper discusses

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the estimation of $m(x)$ when it is assumed that $m(x) = \alpha + \beta' x$ for some α, β. The consistency and asymptotic normality of the estimators (of α and β) are established. Also, a method is given to test the linearity of the regression function $m(x)$.

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1. INTRODUCTION

A number of important recent advances in econometric theory are related to the methods of truncated regression model — the regression model in which the range of the dependent variable is restricted to some interval of $(-\infty,\infty)$, usually the non-negative half-line, such as the income of an individual. Powell [6], [7] used the L_1-norm criterion with some modifications in estimating the regression coefficients in truncated linear models. He proved the consistency and asymptotic normality of his estimates under a set of conditions. On the other hand, Nawata's paper [5] uses the ordinary L_2-norm (least square) criterion, along with a grouping and adjustment of the observed data. In his view, his method has the merit of easy computation compared with the method of Powell.

In this paper we borrow the basic idea of Nawata in grouping and adjusting the observed data. But we shall make simplifications in the procedure of grouping, which enables us to make substantial extensions of the results of [5] under weakened conditions.

2. ESTIMATION OF PARAMETERS IN NON-TRUNCATED CASE

2.1. Assumption of the Model

Let $(X_1,Y_1), \ldots, (X_n,Y_n)$ be iid. samples drawn from a $R^d \times R^1$-valued random variable (X,Y). Denote by $m(x)$ the median of the conditional distribution of Y given $X = x$. We suppose that the conditional distribution function has a form

$$P(Y < y|X = x) = F(y - m(x)) \tag{2.1}$$

where F is a fixed distribution function which is not assumed to be known. Under this assumption we can give Y_i a convenient expression as follows:

$$Y_i = m(X_i) + e_i, \qquad i = 1,\ldots,n \tag{2.2}$$

where $e_1, \ldots, e_n$ are iid. with common distribution F, and $X_1, \ldots, X_n$, $e_1, \ldots, e_n$ are mutually independent. The probability measure of X will be denoted by μ. In this section we make the following assumption concerning F and μ. Further assumptions will be introduced when needed.

1°. $F(0) = 1/2$, $f(x) = F'(x)$ exists in some neighborhood of 0, $f(0) > 0$ and $f'(0)$ exists.

2°. $V = \mathrm{COV}(X)$ exists, and $V > 0$.

3°. μ has no singular component. If μ has an absolute continuous component with density $g(x)$, then for sufficiently small $a > 0$, there exists an open set G_a such that the symmetric difference between G_a and $\{x: g(x) > a\}$ has Lebesgue measure zero.

In this section we assume that the median-regression function $m(x)$ has a linear form

$$m(x) = \alpha + \beta' x \tag{2.3}$$

and the problem is to estimate the parameters α, β, using the samples (X_i, Y_i), $i = 1,$

We shall use $\|a\|$ to denote the Euclidean length of vector a, and $a^{(u)}$ to denote the u-th coordinate of a. If A is a vector or matrix, we use $|A|$ to denote the maximum of the absolute values of the elements of A.

2.2. *The Main Result of Section 2*

Choose $\varepsilon_1 \in (0, \frac{1}{2d})$, $\varepsilon_2 \in (\frac{1}{2}, 1 - d\varepsilon_1)$, $\ell_n = n^{-\varepsilon_1}$, $c_0 > 0$. Decompose R^d into a set J_n^* of supercubes having the form:

$$\{(x^{(1)},\ldots,x^{(d)}): a_i\ell_n \le x^{(i)} < (a_i + 1)\ell_n,\ i = 1,\ldots,d\}.$$

$$a_i = 0, \pm 1, \pm 2, \quad i = 1,\ldots,d. \tag{2.4}$$

For $J \in J_n^*$, use $\#(J)$ to denote the number of elements in the set $J \cap \{X_1,\ldots,X_n\}$. Write

$$\{J\colon J \in J_n^*,\ \#(J) \ge c_0 n^{\varepsilon_2}\} = \{J_{n1},\ldots,J_{nc_n}\} \tag{2.5}$$

We have

$$c_n \le c_0^{-1} n^{1-\varepsilon_2} \le n^{d\varepsilon_1 - \varepsilon'} \tag{2.6}$$

for some $\varepsilon' > 0$, when n is large. Further, write

$$J_{ni} \cap \{X_1,\ldots,X_n\} = \{X_{ni}(1),\ldots,X_{ni}(n_i)\}.$$

By definition,

$$n_i \ge c_0 n^{\varepsilon_2}, \quad i = 1,\ldots,c_n. \tag{2.7}$$

We shall write $Y_{ni}(j)$ and $e_{ni}(j)$ for Y_k and e_k, when $X_{ni}(j) = X_k$. Put

$$X_{ni} = \sum_{j=1}^{c_n} X_{ni}(j)/n_i$$

$$Y_{ni} = \text{med}(Y_{ni}(1),\ldots,Y_{ni}(n_i))$$

$$e_{ni} = \text{med}(e_{ni}(1),\ldots,e_{ni}(n_i))$$

$$N_n = n_1 + n_2 + \ldots + n_{c_n}$$

$$\overline{X}_n = \sum_{i=1}^{c_n} n_i X_{ni}/N_n, \quad \overline{Y}_n = \sum_{i=1}^{c_n} n_i Y_{ni}/N_i, \quad \overline{e}_n = \sum_{i=1}^{c_n} n_i e_{ni}/N_n$$

$$X_{(n)} = (X_{ni}-\overline{X}_n,\ldots,X_{c_n}-\overline{X}_n)', \quad Y_{(n)} = (Y_{ni},\ldots,Y_{nc_n})', \quad e_{(n)} = (e_{ni},\ldots,e_{nc_n})'$$

$$W_n = \text{diag}(n_i,\ldots,n_{c_n}), \qquad P_n = X'_{(n)} W_n X_{(n)}.$$

Define

$$\tilde{\beta}_n = \beta + P_n^{-1} X'_{(n)} W_n e_{(n)}, \quad \tilde{\alpha}_n = \alpha + \overline{X}'_n(\beta - \tilde{\beta}_n) + \overline{e}_n \tag{2.8}$$

and $(\hat{\alpha}_n^{(k)}, \hat{\beta}_n^{(k)})$, $k = 0, 1, \ldots$, by the following induction process. Set

$$\hat{\beta}_n^{(0)} = P_n^{-1} X'_{(n)} W_n Y_{(n)}, \quad \hat{\alpha}_n^{(0)} = \overline{Y}_n - \overline{X}'_n \hat{\beta}_n^{(0)} \tag{2.9}$$

which is the solution of the weighted least squares problem.

$$\sum_{i=1}^{c_n} n_i (Y_{ni} - \alpha - X'_{ni}\beta)^2 = \min!.$$

Suppose that $\hat{\beta}_n^{(k)}$ and $\hat{\alpha}_n^{(k)}$ have already been defined. Put

$$Y_{ni}^{(k+1)}(j) = Y_{ni}(j) - (X_{ni}(j) - X_{ni})'\hat{\beta}_n^{(k)}, \qquad j = 1,\ldots,n_i \tag{2.10}$$

$$Y_{ni}^{(k+1)} = \mathrm{med}(Y_{ni}^{(k+1)}(j): j = 1,\ldots,n_i) \tag{2.11}$$

$$\overline{Y}_n^{(k+1)} = \sum_{i=1}^{c_n} n_i Y_{ni}^{(k+1)}/N_n$$

$$Y_{(n)}^{(k+1)} = (Y_{n1}^{(k+1)},\ldots,Y_{nc_n}^{(k+1)})$$

and then define

$$\hat{\beta}_n^{(k+1)} = P_n^{-1}X'_{(n)}W_nY_{(n)}^{(k+1)}, \quad \hat{\alpha}_n^{(k+1)} = \overline{Y}_n^{(k+1)} - \overline{X}'_n\hat{\beta}_n^{(k+1)} \tag{2.12}$$

which is no other than the solution of the weighted least squares problem

$$\sum_{i=1}^{c_n} n_i(Y_{ni}^{(k+1)} - \alpha - X'_{ni}\beta)^2 = \min!.$$

The $Y_{ni}^{(k+1)}(j)$'s, defined in (2.10), is an "adjustment" of the original observation $Y_{ni}(j)$ of the dependent variable Y. For if we know β, we would set $Y^*_{ni}(j) = Y_{ni}(j) - (X_{ni}(j) - X_{ni})'\beta$, and get the exact model $Y^*_{ni} = \alpha + X'_{ni}\beta + e_{ni}$, $i = 1, \ldots, c_n$. This kind of adjustment was introduced by Nawata [5], who used it to make a "first stage" estimate of α, β, which are used to form a "second stage" estimate of α, β, in case that the dependent variable Y is trucated. We shall use this idea in the next section also. The present work differs from that of Nawata's in some important respects. First, the decomposition of the range of independent variable is greatly simplified, and the conditions imposed on this decomposition is very simple, as compared with the very complicated one introduced by Nawata. Second, we allow the number of sets in the decomposition to go to infinity, which is conceptually reasonable and enables us to reach the optimal covariance matrix of the limit distribution. Third, we do not assume that the range of the independent variable is bounded. Fourth, the number of iterations in our iterative process has a predetermined bound (see Theorem 1 below), while in [5] this number is indefinite. From a practical point of view, it is not reasonable to define an "estimate" by infinite number of iterations.

Now we state the main theorem of this section:

THEOREM 1. Choose an integer r such that

$$r\varepsilon_1 \leq 1/2 < (r+1)\varepsilon_1. \tag{2.13}$$

Then under the conditions stated in Section 1, we have

$$\sqrt{n}\left\{\begin{pmatrix}\hat{\alpha}_n^{(r+1)}\\ \hat{\beta}_n^{(r+1)}\end{pmatrix} - \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\} \xrightarrow{L} N(0,\Lambda^{-1}/4f^2(0)) \tag{2.14}$$

$$|\hat{\alpha}_n^{(r+1)} - \tilde{\alpha}_n| = 0_p(n^{-1/2-\varepsilon_1}) = |\hat{\beta}_n^{(r+1)} - \tilde{\beta}_n| \tag{2.15}$$

where $\Lambda = (\lambda_{ij})$ is a $(d+1) \times (d+1)$ matrix, with

$$\lambda_{oo} = 1, \quad \lambda_{oj} = \lambda_{jo} = EX^{(j)}, \quad \lambda_{ij} = E(X^{(i)}X^{(j)}), \quad i, j = 1,\ldots,d.$$

(2.14) means that, as an estimator of (α,β), $(\hat{\alpha}_n^{(r+1)}, \hat{\beta}_n^{(r+1)})$ possesses an asymptotically optimal covariance matrix.

2.3. A Lemma

The proof of Theorem 1 depends on a limiting theorem concerning the linear forms of $\{e_{ni},\ldots,e_{nc_n}\}$, which we consider separately in this subsection.

LEMMA 1. Let $c_1, c_2, \ldots$ be natural numbers such that

$$\lim_{n\to\infty} c_n/\sqrt{n} = 0. \tag{2.16}$$

For each n, give a set of iid. variables $\{e_{ij}^{(n)}:\ j=1,\ldots,n_i,\ i=1,\ldots,c_n\}$. Here

$$n_1 + n_2 + \ldots + n_{c_n} \leq n \tag{2.17}$$

$$\lim_{n\to\infty} (\sqrt{n}\log n)/\min(n_1,\ldots,n_{c_n}) = 0. \tag{2.18}$$

Assume that the distribution function F of $e_{11}^{(n)}$ does not depend on n, and F satisfies condition 1° of Section 2.1. Let $a_{ni}(j)$: $i=1,\ldots,c_n$, $j=1,\ldots,r$ be constants satisfying the following conditions:

$$\sum_{i=1}^{c_n} n_i a_{ni}(j) = 0, \qquad j = 1,\ldots,r, \quad n = 1,2,\ldots \tag{2.19}$$

$$\lim_{n\to\infty} \sum_{i=1}^{c_n} n_i a_{ni}(j_1) a_{ni}(j_2)/n = \lambda_{j_1 j_2} \tag{2.20}$$

exists and finite for $j_1, j_2 = 1,\ldots,r$.

Define $e_i^{(n)} = \mathrm{med}(e_{i1}^{(n)},\ldots,e_{ic_n}^{(n)})$, $i = 1,\ldots,c_n$, and

$$\xi_{nj} = \sum_{i=1}^{c_n} n_i a_{ni}(j) e_i^{(n)}/\sqrt{n}, \quad j = 1,\ldots,r, \quad \xi_n = (\xi_{n1},\ldots,\xi_{nr})'. \tag{2.21}$$

Then we have

$$\xi_n \xrightarrow{L} N_r(0, \Lambda/4f^2(0)) \tag{2.22}$$

as $n \to \infty$, where Λ is the matrix with elements $\lambda_{j_1 j_2}$.

Proof. Consider first the case $r = 1$, and write for simplicity $a_{ni}(1) = a_{ni}$, $\xi_{n1} = \xi_n$, $\lambda_{11} = \sigma^2$.

Given $\delta > 0$. By the assumption made on F, we have $F(\delta) > 1/2$. Using an inequality of Hoeffding [4], we get

$$P(e_i^{(n)} \geq \delta) \leq P(|\frac{1}{n_i} \sum_{j=1}^{n_i} I(e_{ij}^{(n)} \geq \delta) - (1 - F(\delta))| \geq F(\delta) - \frac{1}{2})$$

$$\leq 2 \exp(- n_i(F(\delta) - 1/2)^2/3).$$

From this and (2.18), we have

$$P(e_i^{(n)} \geq \delta) \leq \exp(- \sqrt{n}), \qquad i = 1,\ldots,c_n$$

for n large. Similarly it is shown that

$$P(e_i^{(n)} \leq -\delta) \leq \exp(- \sqrt{n}), \qquad i = 1,\ldots,c_n$$

for n large. Hence for n_0 large we have

$$\sum_{n=n_0}^{\infty} \sum_{i=1}^{c_n} P(|e_i^{(n)}| \geq \delta) \leq \sum_{n=n_0}^{\infty} \sqrt{n}\, e^{-\sqrt{n}} < \infty.$$

Therefore, wp1 (with probability one) we have

$$|e_i^{(n)}| \leq \delta, \qquad i = 1,\ldots,c_n \tag{2.23}$$

for n large.

Denote by $\{U_{ij}: i=1,2,\ldots,\ j=1,2,\ldots\}$ a family of iid. random variables with common distribution $R(0,1)$, and

$$U_i^{(n)} = \text{med}(U_{i1},\ldots,U_{in_i}), \qquad i = 1,\ldots,c_n.$$

By assumption on F, the inverse function F^{-1} exists in some neighborhood of 1/2, so we can find some $\delta > 0$ such that the distribution functions of $F^{-1}(U_i^{(n)})$ and $e_i^{(n)}$ coincide on $(-\delta,\delta)$. From this and (2.23), it is seen that the assertion

$$\xi_n \xrightarrow{L} N(0, \sigma^2/4f^2(0)) \tag{2.24}$$

is equivalent to

$$\tilde{\xi}_n \triangleq \sum_{i=1}^{c_n} n_i a_{ni} F^{-1}(U_i^{(n)})/\sqrt{n} \xrightarrow{L} N(0, \sigma^2/4f^2(0)). \tag{2.25}$$

According to a theorem of Csörgo and Revesz concerning the strong approximation of quantile process (see [2]) there exist independent N(0, 1/4) random variables $\eta_{ni}, \ldots, \eta_{nc_n}$, such that

$$P(|\sqrt{n_i}|(U_i^{(n)} - \tfrac{1}{2}) - \eta_{ni}| \geq n_i^{-1/2}(A \log n_i + Z)) \leq Be^{-cZ}, \text{ for } |Z| \leq D\sqrt{n_i} \qquad (2/26)$$

where A, B, C, D are positive absolute constants. Choose $Z = 5 \log n_i/c$ and put $K_1 = A + 5/c$, we have

$$\sum_{n=n_0}^{\infty} \sum_{i=1}^{c_n} P(|\sqrt{n_i}|(U_i^{(n)} - \tfrac{1}{2}) - \eta_{ni}| \geq K_1 n_i^{-1/2} \log n_i) \leq B \sum_{n=n_0}^{\infty} \sqrt{n}\, n^{-5/2} < \infty.$$

Therefore, wpl we have

$$|U_i^{(n)} - (\tfrac{1}{2} + \eta_{ni}/\sqrt{n_i})| \leq K_1 n_i^{-1} \log n_i, \qquad i = 1, \ldots, c_n \tag{2.27}$$

for n large. From this it follows that (2.25) is equivalent to

$$\xi_n^* \triangleq \sum_{i=1}^{c_n} n_i a_{ni} F^{-1}(\tfrac{1}{2} + \eta_{ni}/\sqrt{n_i} + \theta_{ni})/\sqrt{n} \xrightarrow{L} N(0, \sigma^2/4f^2(0)) \tag{2.28}$$

where θ_{ni}, $i = 1, \ldots, c_n$ are random variables such that

$$|\theta_{ni}| \le K_1 n_i^{-1} \log n_i, \quad i = 1,\ldots,c_n, \quad n = 1,2,\ldots . \tag{2.29}$$

Since $2\eta_{ni} \sim N(0,1)$, it is well known that (see [3], page 131)

$$P(|\eta_{ni}|/\sqrt{n_i} \ge \varepsilon) \le 2\frac{1}{\sqrt{2\pi}\,2\sqrt{n_i}\,\varepsilon}\exp(-\tfrac{1}{2}(2\sqrt{n_i}\,\varepsilon)^2) \le e^{-\sqrt{n}}$$

for $i = 1, \ldots, c_n$ and large n. Hence we have for large n_0

$$\sum_{n=n_0}^{\infty}\sum_{i=1}^{c_n} P(|\eta_{ni}|/\sqrt{n_i} \ge \varepsilon) \le \sum_{n=n_0}^{\infty} \sqrt{n}\, e^{-\sqrt{n}} < \infty$$

which implies that wpl we have

$$|\eta_{ni}|/\sqrt{n_i} \le \varepsilon, \qquad i = 1,\ldots,c_n \tag{2.30}$$

for n large. Considering (2.29), (2.30), and the assumption made on F, we get

$$F^{-1}(\tfrac{1}{2} + \eta_{ni}/\sqrt{n_i} + \theta_{ni}) = \frac{1}{f(0)}(\eta_{ni}/\sqrt{n_i} + \theta_{ni}) \tag{2.31}$$

$$+ \tfrac{1}{2}(r + \varepsilon_{ni})(\eta_{ni}/\sqrt{n_i} + \theta_{ni})^2$$

where $r = -f'(0)/(f(0))^3$, and $\varepsilon_{ni}, \ldots, \varepsilon_{nc_n}$ are random variables such that

$$\lim_{n\to\infty} \max(|\varepsilon_{ni}|,\ldots,|\varepsilon_{nc_n}|) = 0, \quad \text{a.s.} \tag{2.32}$$

From (2.31), we can rewrite (2.28) as follows:

$$\xi_n^* = T_{n1} + \ldots + T_{n5} \tag{2.33}$$

where

$$T_{n1} = \sum_{i=1}^{c_n} \sqrt{n_i}\, a_{ni} \eta_{ni} / \sqrt{n} f(0)$$

$$T_{n2} = \sum_{i=1}^{c_n} n_i a_{ni} \theta_{ni} / \sqrt{n} f(0)$$

$$T_{n3} = \sum_{i=1}^{c_n} \frac{1}{2}(r + \varepsilon_{ni}) a_{ni} \eta_{ni}^2 / \sqrt{n} f(0)$$

$$T_{n4} = \sum_{i=0}^{c_n} (r + \varepsilon_{ni}) n_i a_{ni} \theta_{ni} \eta_{ni} / \sqrt{n} f(0)$$

$$T_{n5} = \sum_{i=0}^{c_n} \frac{1}{2}(r + \varepsilon_{ni}) \theta_{ni}^2 n_i a_{ni} / \sqrt{n} f(0).$$

Since $\sum_{i=1}^{c_n} n_i a_{ni}^2 / n \to \sigma^2$, we have

$$T_{n1} \xrightarrow{L} N(0, \sigma^2 / 4f^2(0). \qquad (2.34)$$

From (2.29), one finds

$$|T_{n2}| \le \sum_{i=1}^{c_n} \frac{n_i}{n} |a_{ni}| \frac{\sqrt{n} \log n_i}{n_i} K_1 / f(0). \qquad (2.35)$$

From (2.17), by Schwartz inequality,

$$\left(\sum_{i=1}^{c_n} \frac{n_i}{n} |a_{ni}| \right)^2 \le \sum_{i=1}^{c_n} \frac{n_i}{n} a_{ni}^2 \sum_{i=1}^{c_n} \frac{n_i}{n} \le \sum_{i=1}^{c_n} \frac{n_i}{n} a_{ni}^2 \to \sigma^2 < \infty.$$

We see that

$$\sup \left\{ \sum_{i=1}^{c_n} n_i |a_{ni}| / n : \ n=1,2,\ldots \right\} \triangleq K_2 < \infty. \qquad (2.36)$$

Also, by (2.18), it is seen that

$$\max\{ \sqrt{n} \log n_i / n_i : \ i=1,\ldots,c_n \} \to 0, \quad (n \to \infty). \qquad (2.37)$$

From (2.35)-(2.37), one gets

$$\lim_{n\to\infty} T_{n2} = 0. \tag{2.38}$$

For T_{n3}, we note that $E(\eta_{ni}^2) = 1/4$, so by (2.18) and (2.36),

$$E(\sum_{i=1}^{c_n} |a_{ni}|\eta_{ni}^2/\sqrt{n} \le \sum_{i=1}^{c_n} |a_{ni}|/\sqrt{n} = \sum_{i=1}^{c_n} \frac{n_i|a_{ni}|}{n} \frac{\sqrt{n}}{n_i} \to 0.$$

Considering this and (2.32), we get

$$T_{n3} \xrightarrow{P} 0, \quad (n \to \infty). \tag{2.39}$$

T_{n4} and T_{n5} can be handled in a similar way, obtaining

$$T_{n4} \xrightarrow{P} 0, \quad T_{n5} \xrightarrow{P} 0, \quad (n \to \infty). \tag{2.40}$$

Now (2.28) follows from (2.33)-(2.35), (2.39), (2.40). This proves the lemma for $r = 1$.

In order to prove the lemma for general r, take arbitrarily constant vector $t = (t_1,\ldots,t_r)'$, then

$$t'\xi_n = \sum_{i=1}^{c_n} n_i a_{ni} e_i^{(n)}/\sqrt{n}$$

where

$$a_{ni} = \sum_{j=1}^{r} t_j a_{ni}(j), \quad i = 1,\ldots,c_n. \tag{2.41}$$

From (2.19) and (2.20), it is readily seen that

$$\sum_{i=1}^{c_n} n_i a_{ni} = 0, \quad n = 1,2,\ldots$$

$$\lim_{n\to\infty} \sum_{i=1}^{c_n} n_i a_{ni}^2/n = t'\Lambda t.$$

Hence, according to the proved result for the case of $r = 1$, we have

$$t'\xi_n \to N(0,\ t'\Lambda t/4f^2(0)).$$

Since this holds true for arbitrarily chosen t, (2.22) follows, and the lemma is proved.

Conditions of the lemma can be somewhat weakened. Also, the lemma can be proved by resorting to classical methods of Central Limit Theorem, but verification of the conditions will be quite complicated.

2.4. *Proof of Theorem 1*

First note the simple fact that if $u_i = u + t_i'g + h_i$, $i = 1, \ldots, k$, then there exists a vector t in the convex hull of $\{t_1,\ldots,t_k\}$, such that $\mathrm{med}(u_1,\ldots,u_k) = u + t'g + \mathrm{med}(h_1,\ldots,h_k)$. Using this fact, one sees that there exists $X_{ni}^* \in J_{ni}$ (X_{ni}^* depends upon X_i, Y_i, $i = 1, \ldots, n$, and α, β) such that

$$Y_{ni} = \alpha + X_{ni}^{*\prime}\beta + e_{ni} = \alpha + X_{ni}'\beta + e_{ni} + (X_{ni}^* - X_{ni})'\beta. \tag{2.42}$$

Therefore, on putting $X_{(n)}^* = (X_{n1}^*,\ldots,X_{nc_n}^*)'$, one verifies that

$$\hat{\beta}_n^{(0)} - \tilde{\beta}_n = P_n^{-1}X_{(n)}'W_n(X_{(n)}^* - X_{(n)})'\beta. \tag{2.43}$$

We have shown in [1] that under the assumption of the present theorem, one has

$$\lim_{n\to\infty} P_n/n = V, \quad \text{a.s.} \tag{2.44}$$

Also, the absolute value of the (u, v) element of $n^{-1}X_{(n)}'W_n(X_{(n)}^* - X_{(n)})'$ does not exceed

$$\sum_{i=1}^{c_n} n_i|X_{ni}^{(u)} - \bar{X}_n^{(u)}||X_{(n)}^* - X_{(n)}|/n \le n^{-\varepsilon_1}\sum_{i=1}^{c_n} n_i|X_{ni}^{(u)} - \bar{X}_n^{(u)}|/n. \tag{2.45}$$

Here we used the obvious fact that $|X^*_{(n)} - X_{(n)}| \le n^{-\varepsilon_1}$. By an argument similar to that used in [1], it can be shown that

$$\lim_{n\to\infty} \sum_{i=1}^{c_n} n_i |X_{ni}^{(u)} - \bar{X}_n^{(u)}|/n = E|X^{(u)} - EX^{(u)}| < \infty, \quad \text{a.s.} \tag{2.46}$$

From (2.43)-(2.46), it is readily seen that for any given $\delta > 0$, there exists (finite constant) m_0 such that

$$P(|\hat{\beta}_n^{(0)} - \tilde{\beta}_n| \le m_0 n^{-\varepsilon_1}) > 1 - \delta \tag{2.47}$$

for n large.

Now it follows from Lemma 1 that

$$\sqrt{n}(\tilde{\beta}_n - \beta) \xrightarrow{L} N(0, V^{-1}/4f^2(0)). \tag{2.48}$$

The argument is as follows. By definition (2.8), and (2.44), one sees that (2.48) is equivalent to

$$n^{-1/2}V^{-1}X'_{(n)}W_n e_{(n)} \xrightarrow{L} N(0, V^{-1}/4f^2(0)). \tag{2.49}$$

Given $X_1, X_2, \ldots$ and consider the conditional distribution of $T_n \triangleq n^{-1/1}V^{-1}X'_{(n)}W_n e_{(n)}$, then this is just the case studied in Lemma 1 with $r = d$, and

$$\begin{pmatrix} a_{n1}(1) & \ldots & a_{nc_n}(1) \\ a_{n1}(2) & \ldots & a_{nc_n}(2) \\ \cdot & \ldots & \cdot \\ a_{n1}(d) & \ldots & a_{nc_n}(d) \end{pmatrix} = V^{-1}X'_{(n)}.$$

It can easily be verified that the conditions of Lemma 1 are met, with

$$\Lambda = \lim_{n\to\infty} V^{-1}X'_{(n)}W_nX_{(n)}V^{-1}/n = V^{-1}VV^{-1} = V^{-1}, \quad \text{a.s.}$$

So wpl (2.49) holds true conditionally given $X_1, X_2, \ldots$, and it still holds true unconditionally. From (2.48) it follows that

$$\sqrt{n}|\tilde{\beta}_n - \beta| = O_p(1). \tag{2.50}$$

Combining (2.47) and (2.50), one sees that there exists $\bar{m}_0$ such that for n large,

$$P(|\hat{\beta}_n^{(0)} - \beta| \le \bar{m}n^{-t_1}) > 1 - \delta, \quad t_1 = \min(\tfrac{1}{2}, \varepsilon_1). \tag{2.51}$$

By (2.42),

$$\bar{Y}_n = \alpha + \bar{X}_n'\beta + \bar{e}_n + (\bar{X}_n^* - \bar{X}_n)'\beta, \quad (\bar{X}_n^* = \sum_{i=1}^{c_n} n_i X_{ni}^*/n).$$

Hence by (2.8) and (2.9)

$$\hat{\alpha}_n^{(0)} - \alpha_n = \bar{X}_n'(\tilde{\beta}_n - \hat{\beta}_n^{(0)}) + (X_n^T - \bar{X}_n)'\beta. \tag{2.52}$$

Since $\bar{X}_n \to EX$ a.s. and $|\bar{X}_n^* - \bar{X}_n| \le n^{-\varepsilon_1}$, from (2.47) and (2.52) we get a constant ℓ_0 such that for large n

$$P(|\hat{\alpha}_n^{(0)} - \tilde{\alpha}_n| \le \ell_0 n^{-\varepsilon_1}) > 1 - \delta. \tag{2.53}$$

Put k = 0 in (2.12), and notice that $Y_{ni}(j) = \alpha + X'_{ni}(j)\beta + e_{ni}(j)$, we get

$$Y_{ni}^{(1)}(j) = X'_{ni}\beta + \alpha + e_{ni}(j) + (X_{ni} - X_{ni}(j))'(\hat{\beta}_n^{(0)} - \beta).$$

Again there exists X_{ni}^{**} in the convex hull of $\{X_{ni} - X_{ni}(j): j=1,\ldots,n_i\}$, such that

$$Y_{ni}^{(1)} = X'_{ni}\beta + \alpha + e_{ni} + X_{ni}^{**\prime}(\hat{\beta}_n^{(0)} - \beta). \tag{2.54}$$

Since $|X^{**}_{ni}| \le n^{-\varepsilon_1}$, from (2.51) and (2.54), it follows by an argument used earlier that there exists m_1 such that for large n

$$P(|\hat{\beta}_n^{(1)} - \tilde{\beta}_n| \le m_1 n^{-(t_1+\varepsilon_1)}) > 1 - \delta. \tag{2.55}$$

Combining this and the fact that $|\tilde{\beta}_n - \beta| = 0_p(n^{-1/2})$, we find $\bar{m}_1$ such that for large n

$$P(|\hat{\beta}_n^{(1)} - \beta| \le \bar{m}_1 n^{-t_2}) > 1 - \delta, \quad t_2 = \min(\tfrac{1}{2}, t_1+\varepsilon_1). \tag{2.56}$$

From (2.8), (2.12) (setting k = 0) and (2.54), one gets

$$\hat{\alpha}_n^{(1)} - \tilde{\alpha}_n = \bar{X}_n^{**'}(\hat{\beta}_n^{(0)} - \beta) - \bar{X}_n'(\hat{\beta}_n^{(1)} - \tilde{\beta}_n). \tag{2.57}$$

From (2.51), (2.55), and the fact that $|\bar{X}^{**}_n| \le n^{-\varepsilon_1}$, we find ℓ_1 such that for any large n

$$P(|\hat{\alpha}_n^{(1)} - \tilde{\alpha}_n| \le \ell_1 n^{-(t_1+\varepsilon_1)}) > 1 - \delta. \tag{2.58}$$

In deriving (2.58) one should also note that, as shown above, the event $\{|\hat{\beta}_n^{(1)} - \tilde{\beta}_n| \le m_1 n^{-(t_1+\varepsilon_1)}\}$ is a consequence of $\{|\hat{\beta}_n^{(0)} - \beta| \le \bar{m} n^{-t_1}\}$.

Continuing this process, one finds generally that there exists constants m_k, $\bar{m}_k$ and ℓ_k, such that for n large we have

$$P(|\hat{\beta}_n^{(k)} - \tilde{\beta}_n| \le m_k n^{-(t_k+\varepsilon_1)}) > 1 - \delta \tag{2.59}$$

$$P(|\hat{\beta}_n^{(k)} - \beta| \le \bar{m}_k n^{-t_{k+1}}) > 1 - \delta \tag{2.60}$$

$$P(|\hat{\alpha}_n^{(k)} - \tilde{\alpha}_n| \le \ell_k n^{-(t_k+\varepsilon_1)}) > 1 - \delta \tag{2.61}$$

with

$$t_{k+1} = \min(\tfrac{1}{2}, t_k + \varepsilon_1).$$

Since $r\varepsilon_1 \leq 1/2$ and $(r + 1)\varepsilon_1 > 1/2$, we have $t_i = i\varepsilon_1$ for $i \leq r$, and so $t_r + \varepsilon_1 = (r + 1)\varepsilon_1$, $t_{k+1} = 1/2$. Therefore, on putting $k = r + 1$ in (2.59) and (2.61), we get (2.15).

In view of (2.15), (2.14) is equivalent to

$$\sqrt{n}\left[\begin{pmatrix} \tilde{\alpha}_n \\ \tilde{\beta}_n \end{pmatrix} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right] \xrightarrow{L} N_{d+1}(0, \Lambda^{-1}/4f^2(0)) \tag{2.62}$$

As $\tilde{\beta}_n$ and $\tilde{\alpha}_n$ are linear functions of $e_{(n)}$, (2.62) can easily be proved by using Lemma 1, the argument is just the same as we employed in showing (2.49). This concludes the proof of the theorem.

The assertion (2.14) still holds true when $r + 1$ in the left hand side of (2.14) is replaced by r, or by some $k > r + 1$. But iterating beyond $(r + 1)$ rounds is non-profitable, in view of the fact that $t_{r+1} = t_{r+2} = \ldots = 1/2$.

3. ESTIMATION OF PARAMETERS IN TRUNCATED CASE

In this section we study the case in which the dependent variable is truncated at zero. If the original values of $\tilde{Y}$ are $\tilde{Y}_1, \ldots, y_n$, then actually we observe

$$Y_i = \tilde{Y}_i I(\tilde{Y}_i > 0), \qquad i = 1,\ldots,n.$$

Introduce J_n^* as we did in Section 2.2. Choose constants $c' > 0$, $\varepsilon' \in (\varepsilon_1, \varepsilon_1, 1)$, where ε_1 has been introduced at the beginning of Section 2.2. Divide J_n^* into three disjoint parts. Let $H_i = \sum_{j=1}^{n_i}((Y_{ni}(j) > 0)$.

$$J_{n1}^* = \{J_{ni} : H_i > n_i/2 + c' n_i^{\varepsilon_1'}, \quad i = 1,\ldots,n\}$$

$$J_{n2}^* = \{J_{ni} : H_i < n_i/2 - c' n_i^{\varepsilon_1'}, \quad i = 1,\ldots,n\}$$

$$J_{n3}^* = J_n^* - (J_{n1}^* \cup J^*_{n2}).$$

For convenience, we shall in this section write $\tilde{x}'\gamma$ for $\alpha + x'\beta$, by introducing $\tilde{x} = (1,x')'$ and $\gamma = (\alpha,\beta')'$. We use x and α to replace $\tilde{x}$ and γ. In this way we change $\alpha + x'\beta$ to $x'\alpha$.

The following lemma will be used in the sequel.

LEMMA 2. wp1 we have for any given $\varepsilon_2' < \varepsilon_1'$.

$$J_{ni} \in J_{ni}^* \Rightarrow X_{ni}'\alpha \geq n_i^{-1+\varepsilon_2'}, \qquad i = 1,\ldots,n \tag{3.1}$$

$$J_{ni} \in J_{n2}^* \Rightarrow X_{ni}'\alpha \leq -n_i^{-1+\varepsilon_2'}, \qquad i = 1,\ldots,n \tag{3.2}$$

for n sufficiently large.

Proof. Assume that $X_{ni}'\alpha < n_i^{-1+\varepsilon_2'}$, then

$$X_{ni}'(j) < n_i^{-1+\varepsilon_2'} + n^{-\varepsilon_1} \leq n_i^{-1+\varepsilon_2'} + n_i^{-\varepsilon_1} \leq n_i^{-1+\varepsilon_0}, \qquad j = 1,\ldots,n_i$$

for some $\varepsilon_0 < \varepsilon_1'$. Hence, in order to have $J_{ni} \in J_{n1}^*$, the inequality

$$H_i \overset{\Delta}{=} \sum_{j=1}^{n_i} I(e_{ni}(j) > -n_i^{-1+\varepsilon_0}) \geq \frac{1}{2} n_i + c' n_i^{\varepsilon_1'}$$

must be true. On the other hand, from the assumptions made on F (see Section 2.1), one can find constant $c'' > 0$ such that

$$p \overset{\Delta}{=} P(e_{ni}(j) > -n_i^{-1+\varepsilon_0}) \leq 1/2 + c'' n_i^{-1+\varepsilon_0}.$$

Using Hoeffding's inequality [2], and abserving that

$$\varepsilon_1 < 1/2 \Rightarrow \varepsilon' > 1 - \varepsilon_1 > 1/2, \quad n_i \geq c_0 n^{\varepsilon_2} \quad \text{(see (2.7))},$$

we get for n large

$$P^*(J_{ni} \in J^*_{n1}) \leq P^*(|H_i/n_i - p| \geq c' n_i^{-1+\varepsilon_1'} - c'' n_i^{-1+\varepsilon_0})$$

$$\leq P^*(|H_i/n_i - p| \geq \frac{1}{2} c' n_i^{-1+\varepsilon_1'}) \leq 2\exp(-n_i(\frac{1}{2} c' n_i^{-1+\varepsilon_1'})^2/3) \leq n^{-3} \qquad (3.3)$$

simultaneously for $i = 1, \ldots, c_n$, where $P^* = P^*(X_1, X_2, \ldots)$ is the conditional distribution given $X_1, X_2, \ldots$. Since (3.3) holds for each $(X_1, X_2, \ldots)$, we get for n large

$$P(J_{ni} \in J^*_{n1}) \leq n^{-3} \qquad (3.4)$$

simultaneously for $i = 1, \ldots, c_n$. Introduce the event

$$E_n = \{\text{for some } i = 1, \ldots, c_n, \; X'_{ni}\alpha < n_i^{-1+\varepsilon_2'} \text{ but } J_{ni} \in J^*_{n1}\}.$$

Then since $c_n \leq n$, we have $P(E_n) \leq c_n/n^3 \leq n^{-2}$, yielding

$$P(E_n \text{ i.o.}) = 0$$

which means that wp1 $X'_{ni}\alpha < n_i^{-1+\varepsilon_2'} \Rightarrow J_{ni} \bar{\in} J^*_{n1}$ for all $i = 1, \ldots, c_n$ and n sufficiently large. This is just (3.1). (3.2) can be proved in a similar fashion.

3.1 Estimation Using Only J^*_{ni} *-cells*

If a cell J_{ni} belongs to J^*_{n1}, then, although the observations of the dependent variable related to this cell might have been truncated, the median of the original observations can still be calculated. Therefore the method of the previous section can be applied to the collection of these cells, yielding an estimate for α.

In order to avoid the introduction of numerous new notations, from now on in this section we shall redefine $J_{n1}, \ldots, J_{nc_n}$ as the elements in J^*_{n1}. Other notations in Section 2, too, are redefined in accordance with this change. For instance, the symbol N_n should be understood as

$$N_n = \sum_{\{i=J_{ni} \in J^*_{n1}\}} n_i .$$

Ending this process we get a redefined estimate of α (the original (α,β')), which we now denote by $\hat{\alpha}_n^{(r+1)}$.

For this estimate the following theorem is true:

THEOREM 2. Suppose in addition to the conditions of Theorem 1 that

$$P(X'\alpha > 0) > 0. \tag{3.5}$$

$$\tilde{V} = \mathrm{COV}(X | X'\alpha > 0) > 0. \tag{3.6}$$

Then, as $n \to \infty$, we have

$$\sqrt{N_n}\,(\hat{\alpha}_n^{(r+1)} - \alpha) \xrightarrow{L} N(0, \tilde{V}^{-1}/4f^2(0)).$$

Proof. On account of Lemma 2, this theorem can be proved by largely the same method employed in proving Theorem 1. So the details are omitted.

3.2 Tobit-Type Estimate

In this subsection, in addition to the cells in J^*_{n1}, use will be made on cells belonging to J^*_{n2} in order to form a Tobit-type estimator for α. It is believed that by so doing we are able to make some improvements on $\hat{\alpha}_n^{(r+1)}$ discussed earlier. As Nawata declared in [5], his simulation results in some cases seem to give support to this belief. Theoretically, the problem is complicated as the probable improvements are likely to depend on actual situations (underlying distributions, sample sizes, method of decomposition of the range of independent variables, etc.) and would be difficult to justify in a reasonably general setting.

Now use $\tilde{J}_{n1}, \ldots, \tilde{J}_{nd_n}$ to denote the cells belonging to J^*_{n2}. The center of $\tilde{J}_{ni}$ will be denoted by $\tilde{\underline{X}}_{ni}$, $i = 1, \ldots, d_n$. Put $m_i = \#(\tilde{J}_{ni})$ (the number of elements in $\tilde{J}_{ni} \cap \{X_1, \ldots, X_n\}$), and

$$L(\alpha,\sigma) = \prod_{i=1}^{d_n} \Phi(-\sqrt{m_i}\underline{X}'_{ni}\alpha/\sigma) \prod_{i=1}^{c_n} \sigma^{-1}\exp[-n_i Y_{ni}^{(r+1)} - X'_{ni}\alpha)^2/2\sigma^2] \tag{3.7}$$

where Φ is the distribution function of $N(0,1)$.

If (α^*_n, σ^*_n) maximizes $L(\alpha,\sigma)$, we use α^*_n as an estimate of α. This kind of estimate was first considered by Tobin [9].

We shall prove the following theorem.

THEOREM 3. Suppose that in addition to the conditions of Theorem 2, we have

$$E|X|^{2+\delta} < \infty \quad \text{for some} \quad \delta > 0.$$

Choose $\varepsilon_1 < \delta/(4+2\delta)$ (see the beginning of Section 2.2), and ε'_2 in (3.1), that

$$\varepsilon'_2 > 1 - \delta/(4+2\delta). \tag{3.8}$$

Then, as $n \to \infty$, we have

$$\sqrt{N_n}(\alpha_n^* - \alpha) \xrightarrow{L} N(0, \tilde{V}/4f^2(0)) \tag{3.9}$$

where $\tilde{V}$ is defined in (3.6).

This theorem indicates that in the asymptotic sense the Tobit-type estimator α_n^* makes no improvement over $\hat{\alpha}_n^{(r+1)}$, which is the ordinary LS estimator based upon only the cells in J_{n1}^*. Needless to say that in practical applications the sample size n may not necessarily be large. In such cases the question remains as to which one is superior over the other.

In defining α_n^* we make no use of those cells which do not belong to $J_{n1}^* \cup J_{n2}^*$. From a practical point of view this poses no serious problem, as we always can choose c_0, ϵ_2, c_1', ϵ_1' small enough to allow the inclusion of more cells. Theoretically speaking, as long as $P(X'\alpha = 0) = 0$ (which is the case when X is non-atomic), the proportion of sample points not used in the definition of α_n^* goes to zero as $n \to \infty$. Nevertheless, it is interesting to ask whether or not it is possible to invent a trick which enables us to use all sample points in the definition of α_n^*, while allowing the number of cells to go to infinity and retains the basic asymptotic property of α_n^* as described in Theorem 3.

The proof of Theorem 3 will be preceeded by several lemmas.

LEMMA 3. Suppose that $\xi_1, \xi_2, \ldots$ is a sequence of iid. random variables, and $E|\xi_1|^a < \infty$ for some $a > 0$. Then

$$\lim_{n\to\infty} n^{-1/a}\max(|\xi_1|,\ldots,|\xi_n|) = 0, \quad \text{a.s.} \tag{3.10}$$

Proof is simple.

LEMMA 4. Denote the residual sum of squares by

$$R_n = \sum_{i=1}^{c_n} (Y_{ni}^{(r+1)} - X'_{ni}\hat{\alpha}_n^{(r+1)})^2 \tag{3.11}$$

Then, under the conditions of Theorem 2, we have wpl

$$\sigma_n^2 \triangleq R_n/C_n \xrightarrow{P^*} \sigma_0^2, \quad (n\to\infty) \tag{3.12}$$

where

$$\sigma_0^2 = (4f^2(0))^{-1} \tag{3.13}$$

$$P^* = P^*(X_1, X_2, \ldots) = \text{the conditional probability measure given } X_1, X_2, \ldots . \tag{3.14}$$

Proof. We proceed to show that WPl there exists random variable $\eta_n \sim \chi^2_{c_n - d}$, such that

$$R_n/\sigma_0^2 - \eta_n - 0_p(\sqrt{c_n}\, n^{-\varepsilon_1}) \xrightarrow{P^*} 0, \quad (n\to\infty). \tag{3.15}$$

From this, (3.12) follows at once.

In order to prove (3.15), we rewrite R_n as

$$R_n = Y_n^{(r+1)\prime}(W_n - W_n X_{(n)} P_n^{-1} X'_{(n)} W_n) Y_n^{(r+1)}. \tag{3.16}$$

Notations involved are defined in Section 2.2. Put

$$Z_{ni} = X'_{ni}\alpha + e_{ni}, \quad i = 1,\ldots,c_n, \quad Z_n = (Z_{n1},\ldots,Z_{nc_n})'.$$

It is not difficult to see by definition (2.10) and Theorem 2 that

$$Y_n^{(r+1)} = Z_n + \xi_n, \qquad \xi_n = (\xi_{n1},\ldots,\xi_{nc_n})' \tag{3.17}$$

where $\xi_{n1}, \ldots, \xi_{nc_n}$ are random variables uniformly (in i) of the order

$O_p(n^{-(1/2+\varepsilon_1)})$ as $n \to \infty$, which means that for arbitrarily given $\varepsilon > 0$, a constant M_ε exists so that for n large

$$P(|\xi_{ni}| \le M_\varepsilon n^{-(1/2+\varepsilon_1)}, \quad i = 1,\ldots,c_n) > 1 - \varepsilon. \tag{3.18}$$

Put

$$R_n = Z_n'(W_n - W_n X_{(n)} P_n^{-1} X_{(n)}' W_n) Z_n. \tag{3.19}$$

Then by exactly the same way as in Theorem 1 of [1], we can show that wpl there exists $\eta_n \sim \chi^2_{c_n - d}$, such that

$$\tilde{R}_n/\sigma_0^2 - \eta_n \xrightarrow{P^*} 0. \tag{3.20}$$

This is true because the strong approximation of e_{ni} in [1] is valid to e_{ni} in this paper also, as we indicated in Lemma 1. Now

$$|R_n - \tilde{R}_n| \le \xi_n' W_n \xi_n + 2\, e_{(n)}'(W_n - W_n X_{(n)} P_n^{-1} X_{(n)}' W_n)\xi_n|. \tag{3.21}$$

From (3.18) we have

$$\xi_n' W_n \xi_n = O_p(n^{-2\varepsilon_1}). \tag{3.22}$$

By Schwartz inequality, writing $Q_n = W_n - W_n X_{(n)} P_n^{-1} S_{(n)} W_n$, we get

$$\begin{aligned}(e_{(n)}' Q_n \xi_n)^2 &\le e_{(n)}' Q_n e_{(n)} \cdot \xi_n' Q_n \xi_n \\ &\le e_{(n)}' Q_n e_{(n)} \cdot \xi_n' W_n \xi_n \\ &= \tilde{R}_n \cdot \xi_n' W_n \xi_n.\end{aligned}$$

From this and (3.20), (3.22), we have

$$(e_{(n)}' Q_n \xi_n)^2 = O_p(c_n n^{-2\varepsilon_1}). \tag{3.23}$$

Now (3.15) follows from (3.20)-(3.23) and Lemma 4 is proved.

LEMMA 5. Under the conditions of Theorem 3, the sequence $\{\sigma_n^*\}$ is bounded in probability.

Proof. First we make an estimate on $L(\hat{\alpha}_n^{(r+1)}, \sigma_n)$. For this purpose, note that by Lemma 2, wpl we have

$$\underline{X}_{ni}'\alpha \le n_i^{-1+\varepsilon_2'}, \quad i = 1,\dots,c_n \tag{3.24}$$

for n large. By Theorem 2, $\hat{\alpha} - \hat{\alpha}_n^{(r+1)} = 0_p(n^{-1/2})$, and by Lemma 3 (considering that $E|X|^{2+\delta} < \infty$) for arbitrarily given $\varepsilon > 0$, we have for n large

$$P(|\underline{X}'_{ni}(\alpha - \hat{\alpha}_n^{(r+1)})| \le n^{-\delta/(4+2\delta)}, i=1,\dots,c_n) \ge 1 - \varepsilon. \tag{3.25}$$

By the choice of ε_2', $-1 + \varepsilon_2' > -\delta/(4+2\delta)$. Hence from (3.24) and (3.25), we have for n large

$$P(\underline{X}_{ni}'\hat{\alpha}_n^{(r+1)} \le -\tfrac{1}{2}m_i^{-1+\varepsilon_2'}, i=1,\dots,c_n) > 1 - \varepsilon. \tag{3.26}$$

Combining this with (3.12), we have for n large

$$P(\underline{X}_{ni}'\hat{\alpha}_n^{(r+1)} \le -\tfrac{1}{2}m_i^{-1+\varepsilon_2'}, i=1,\dots,c_n; \sigma_n \le 2\sigma_0) > 1 - \varepsilon. \tag{3.27}$$

Since $m_i \ge c_0 n^{\varepsilon_2}$ (see (2.7)), and $\varepsilon_2' > 1 - \delta/(4+2\delta) > \frac{1}{2}$, we have

$$a \overset{\Delta}{=} 1 - 2(1 - \varepsilon_2') > 0$$

and

$$m_i(m_i^{-1+\varepsilon_2'})^2 = m_i^a \ge c_0 n^{\varepsilon_2 a}, \quad i = 1,\dots,d_n.$$

Since $\Phi(t) \ge 1 - (\sqrt{2\pi}\,t)^{-1}\exp(-t^2/2)$ for $t > 0$, and $\log(1-\chi) > -2\chi$ for $\chi > 0$ sufficiently small. We see that, in case the event appearing in the left hand side of (3.27) occurs, we have

$$\log \prod_{i=1}^{d_n} \Phi(-\sqrt{m_i}\underline{X}_{ni}'\hat{\alpha}_n^{(r+1)}/\sigma_n) \geq -2d_n \exp(-c_0 n^{\varepsilon 2^a}/8\sigma_0^2)/(\sqrt{2\pi c_0 n^{\varepsilon 2^a}}\,\sigma_0)$$

$$\geq -n^{-k} \to 0, \quad \text{as } n \to \infty \text{ for any } k > 0. \tag{3.28}$$

Therefore, for arbitrarily given $\varepsilon > 0$, when n is sufficiently large, we have

$$P(L(\hat{a}_n^{(r+1)}, \sigma_n) \geq \frac{1}{2}\sigma_n^{-c_n} e^{-c_n/2}) > 1 - \varepsilon. \tag{3.29}$$

But if $\sigma > \sqrt{e}\,\sigma_n$, we shall have

$$L(\alpha, \sigma) \leq \sigma^{-c_n} < \frac{1}{2}\sigma_n^{-c_n} e^{-c_n/2}$$

for any α and n large. From this fact and (3.29), we see that

$$P(\sigma_n' < 2\sqrt{e}\,\sigma_n) > 1 - \varepsilon \tag{3.30}$$

for n large, and this concludes the proof of the lemma.

Now we can prove Theorem 3. Given $\varepsilon > 0$, for any α_0 with $\|\alpha_0 - \hat{\alpha}_n^{(r+1)}\| \geq \varepsilon/\sqrt{n}$, we have

$$\log L(\alpha_0, \sigma_n^*) \leq -n \log \sigma_n^* - \frac{1}{2\sigma_n^{*2}} \sum_{i=1}^{c_n} (Y_{ni}^{(r+1)} - X_{ni}'\alpha_0)^2$$

$$= -n \log \sigma_n^* - R_n/2\sigma_n^{*2} - (\alpha_0 - \hat{\alpha}_n^{(r+1)})' P_n (\alpha_0 - \hat{\alpha}_n^{(r+1)})'.$$

We recall that $P_n = X_{(n)}' W_n X_{(n)}$. Since $P_n/n \to \tilde{\Lambda} = \mathrm{COV}(X \mid X'\alpha > 0) > 0$, we get wpl for n large

$$\log L(\alpha_0, \sigma_n^*) \leq -n \log \sigma_n^* - R_n/2\sigma_n^{*2} - \underline{\lambda}\varepsilon^2/2 \tag{3.31}$$

simultaneously for all α_0 such that $\|\alpha_0 - \alpha_n^{(r+1)}\| > \varepsilon/\sqrt{n}$, where $\underline{\lambda} > 0$ is the smallest eigenvalue of $\tilde{\Lambda}$.

On the other hand, (3.28) still holds true when σ_n is replaced by any $\sigma' > 0$. The convergence to zero would be uniform for $\sigma' \le 2\sigma_0$, in case that the event appearing in the left hand side occurs. Therefore, in cases that the events appearing in the left hand side of (3.26) and (3.30) both occur, we shall have

$$\log L(\hat{\alpha}_n^{(r+1)}, \sigma_n^*) \ge -\log \sigma_n^* - R_n/2\sigma_n^{*2} - \varepsilon_n \tag{3.32}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$. From (3.31) and (3.32), we get

$$P(\sup\{L(\alpha_0, \sigma_n^*): \|\alpha_0 - \hat{\alpha}_n^{(r+1)}\| \ge \varepsilon/\sqrt{n}\} < L(\hat{\alpha}_n^{(r+1)}, \sigma_n^*)) > 1 - 2\varepsilon$$

for n large. This implies that

$$P(\|\alpha_n^* - \hat{\alpha}_n^{(r+1)}\| \ge \varepsilon/\sqrt{n}) < 2\varepsilon$$

for n large. Therefore

$$\sqrt{n}(\alpha_n^* - \hat{\alpha}_n^{(r+1)}) \xrightarrow{P} 0, \quad (n \to \infty). \tag{3.33}$$

Now (3.9) follows from Theorem 2 and (3.33). This concludes the proof of Theorem 3.

3.3 Estimation of σ_0^2

Under the method of estimation of the present paper, from a large-sample point of view, σ_0^2 defined in (3.13) plays the role of error variance.

similar to the case of α, we can define two estimates of σ_0^2. One is σ_n^2, which uses only those cells in J_{n1}^* and is the common estimate of error variance based on the residual sum of squares. Another is σ_n^{*2}, which is a kind of maximum likelihood estimate in the Tobit model. The following lemma reveals that these two are asymptotically equivalent.

LEMMA 6. Under the condition of Theorem 3, we have

$$n^a(\sigma_n^* - \sigma_n) \to 0, \quad \text{a.s.} \quad (n \to \infty) \tag{3.34}$$

for any constant $a > 0$.

Proof. By Lemma 5, (3.28), we have wpl

$$\log \prod_{i=1}^{d_n} \Phi(-\sqrt{m_i}\, X'_{ni}\alpha_0/\sigma_n^*) - \log \prod_{i=1}^{d_n} \Phi(-\sqrt{m_i}\, X_{ni}\hat{\alpha}_n^{(r+1)}/\sigma_n) \le n^{-k} \tag{3.35}$$

for n large, where k is arbitrarily given. Further

$$\begin{aligned} T_n &\overset{\Delta}{=} \log \prod_{i=1}^{c_n} \sigma_n^{*-1} \exp[-n_i(Y_{ni}^{(r+1)} - X'_{ni}\alpha_n^*)^2/2\sigma_n^{*2}] \\ &\quad - \log \prod_{i=1}^{c_n} \sigma_n^{-1} \exp[-n_i(Y_{ni}^{(r+1)} - X'_{ni}\hat{\alpha}_n^{(r+1)})^2/2\sigma_n^2] \\ &= -\frac{1}{2\sigma_n^{*2}} \sum_{i=1}^{c_n} n_i(Y_{ni}^{(r+1)} - X'_{ni}\alpha_n^*)^2 + \frac{1}{2\sigma_n^2} \sum_{i=1}^{c_n} n_i(Y_{ni}^{(r+1)} - X'_{ni}\hat{\alpha}_n^{(r+1)})^2 \\ &\quad - c_n \log \sigma_n^* + c_n \log \sigma_n . \end{aligned}$$

Since

$$\sum_{i=1}^{c_n} n_i(Y_{ni}^{(r+1)} - X'_{ni}\alpha_n^2)^2 \ge \sum_{i=1}^{c_n} n_i(Y_{ni}^{(r+1)} - X'_{ni}\hat{\alpha}_n^{(r+1)})^2 = R_n,$$

we have

$$\begin{aligned} T_n &\le R_n(\sigma_n^{*2} - \sigma_n^2)/(2\sigma_n^2\sigma_n^{*2}) - c_n \log(\sigma_n^*/\sigma_n) \\ &= c_n[(1 - x^2)/2 + \log x] \le -c_n|x - 1|^2/2 \end{aligned} \tag{3.36}$$

where $x = \sigma_n/\sigma_n^*$. Hence, if $|\sigma_n/\sigma_n^* - 1| \ge \epsilon n^{-a}$, then, by (3.35) and (3.36), we shall have, on taking $k = 2a + 1$ in (3.35), that

$$\log L(\alpha_n^*, \sigma_n^*) - \log L(\hat{\alpha}_n^{(r+1)}, \sigma_n) < 0 \tag{3.37}$$

for n large. But (3.37) is impossible as (α_n^*, σ_n^*) maximize $L(\alpha, \sigma)$. This shows that wpl we have

$$|n^a(\sigma_n^* - \sigma_n)| < \varepsilon$$

for n large, and (3.34) is proved.

THEOREM 4. Under the conditions of Theorem 3:

1°. If X is purely atomic with c distinct atoms, $d < c < \infty$, then as $n \to \infty$

$$\sigma_n^2/\sigma_0^2 \xrightarrow{L} \chi^2_{x-d}, \qquad \sigma_n^*/\sigma_0^2 \xrightarrow{L} \chi^2_{c-d}. \tag{3.38}$$

2°. In other cases we have as $n \to \infty$

$$\sqrt{c_n}(\sigma_0^2 - \sigma_0^2)/\sqrt{2} \xrightarrow{L} (N, 0) \tag{3.39}$$

$$\sqrt{c_n}(\sigma_n^{*2} - \sigma_0^2)/\sqrt{2} \xrightarrow{L} (N, 0) \tag{3.40}$$

and

$$\sqrt{2c_n\sigma_n^2/\sigma_0^2} - \sqrt{2(c_n - d)} \xrightarrow{L} N(0, 1) \tag{3.41}$$

$$\sqrt{2c_n\sigma_n^{*2}/\sigma_0^2} - \sqrt{2(c_n - d)} \xrightarrow{L} N(0, 1). \tag{3.42}$$

Proof. In case 1° we have wpl $c_n = c$ for n large. By (3.15), wpl, under P* we have $\sigma_n^2/\sigma_0^2 \xrightarrow{L} \chi^2_{c-d}$. Hence this is also true unconditionally. This proves the first assertion of (3.38). The second follows from the first and Lemma 6.

In case 2° we have $c_n \to \infty$, a.s. From (3.15) and the central limit theorem, wpl, under P* we have (3.39). So (3.39) is still true unconditionally. (3.40) follows from (3.39) and Lemma 6.

(3.41) follows from (3.15), and the following two facts:

a) if $\xi_n \sim \chi_n^2$, then $\sqrt{2\xi_n} - \sqrt{2n} \xrightarrow{L} N(0,1)$, as $n \to \infty$,

b) $\sqrt{x + a(x)} - \sqrt{x} \to 0$, as $x \to \infty$ and $\lim\limits_{x\to\infty} a(x)/\sqrt{x} = 0$.

(3.42) follows from (3.34) and (3.41).

4. TESTING OF LINEARITY

In practical applications we are often not sure that the regression function (the conditional median of Y given X) is linear, and a test for this hypothesis is desirable. In this section we shall propose such a test.

The idea behind the test is quite simple and is similar to the one proposed in [1], where the regression function is defined as $E(Y|X=x)$ and no truncation is allowed. From now on we use H_0 to denote the linear hypothesis (2.3).

If (2.3) is not rure, then the residual sum of squares R_n, defined by (3.11), tends to become larger. Therefore a reasonable test of H_0 is to reject it when

$$R_n > C \tag{4.1}$$

for some C, and accept it otherwise. C is chosen according to the preassigned size α_0. In order to do this, we have to find an estimate $\tilde{\sigma}_n^2$ of $\sigma_0^2 = (1/4f^2(0))$ such that (3.15) still holds true when σ_0^2 is replaced by $\tilde{\sigma}_n^2$, under H_0. For if such an estimate $\tilde{\sigma}_n^2$ has been found, then (3.41) remains valid when σ_0^2 is replaced by $\tilde{\sigma}_n^2$ (under H_0), and we can choose

$$C = \tilde{\sigma}_n^2(\sqrt{2(c_n - d)} + u_{\alpha_0})^2/2 \tag{4.2}$$

where u_{σ_0} is defined by $\Phi(u_{\alpha_0}) = 1 - \alpha_0$. The test (4.1) is asymptotically similar with size α_0.

The problem of estimating σ_0^2 is reduced to the problem of f(0), the value of the density function of e_i at zero.

It is easy to see that if an estimate $\tilde{\sigma}_n^2$ of σ_0^2 satisfies

$$\sqrt{c_n}(\tilde{\sigma}_n^2 - \sigma_0^2) \xrightarrow{P} 0. \tag{4.3}$$

Then $\tilde{\sigma}_n^2$ will have the property required in Section 4.1. It is obvious that if we can find an estimate $f_n(0)$ of $f(0)$ such that

$$\sqrt{c_n}(f_n(0) - f(0) \xrightarrow{P} 0, \tag{4.4}$$

then $\tilde{\sigma}_n^2 \triangleq (4f_n(0))^{-1}$ satisfies (4.3).

Choose $\varepsilon_1 \in (0, 1/3d)$ in the definition of J_n^* in Section 2.2. Since $\varepsilon_1 < 1/3$, we have $\varepsilon_1' > 2/3$ in the definition of J_{n1}^*. Take $\varepsilon_2' > 1 - \varepsilon_1$ in (3.1), then $1 - \varepsilon_2' < 1/3$.

Choose $\varepsilon_0 \in (0, (\varepsilon_2 - \frac{2}{3})/4)$ (see (2.5)) and $c_0 > 0$. Select out such cells I in J_{1n}^* satisfying the condition

$$x \in I \Rightarrow |x| \le c_0 n^{\varepsilon_0}. \tag{4.5}$$

For convenience we shall denote all these cells by $J_{n1}, \ldots, J_{nc_n'}$.

Define

$$I_{ni} = \{j: |Y_{ni}(j) - X_{ni}'(j)\hat{\alpha}_n^{(r+1)}| < n^{-1/3},\ j = 1, \ldots, n_i\}, \quad i = 1, \ldots, c_n'.$$

Since

$$X_{ni}'(j)\hat{\alpha}_n^{(r+1)} = (X_{ni}'(j) - X_{ni}')\hat{\alpha}_n^{(r+1)} + X_{ni}'(\hat{\alpha}_n^{(r+1)} - \alpha) + X_{ni}'\alpha \tag{4.6}$$

and from Theorem 2 we have

$$\hat{\alpha}_n^{(r+1)} = 0_p(1), \qquad |\hat{\alpha}_n^{(r+1)} - \alpha| = 0_p(n^{-1/2}).$$

Also, $|X_{ni}'(j) - X_{ni}'| \le n^{-\varepsilon_1}$, $|X_{ni}'| \le c_0 n^{\varepsilon_0}$ for $i = 1, \ldots, c_n'$, and by Lemma 2, wp1 $X_{ni}'\alpha \ge n^{-1+\varepsilon_2'}$, $i = 1, \ldots, c_n$ for n large. We see from (4.6) that

$$\lim_{n\to\infty} P(E_n) = 1 \tag{4.7}$$

where E_n is the event

$$E_n = \{j \in I_{ni} \text{ for some } i = 1,\ldots,c_n' \Rightarrow Y_{ni}(j) > 0\}. \tag{4.8}$$

When E_n occurs, the number of elements g_{ni} in I_{ni} can be calculated from the truncated observations of the dependent variable Y, and the quantity

$$g_n \overset{\Delta}{=} \sum_{i=1}^{c_n'} g_{ni}$$

is well defined in E_n (can be calculated from the truncated samples when E_n occurs).

Since

$$|(Y_{ni}(j) - X_{ni}'(j)\hat{\alpha}_n^{(r+1)}) - e_{ni}(j)| \le |X_{ni}(j)||\hat{\alpha}_n^{(r+1)} - \alpha|d,$$

there exists constant A such that

$$\lim_{n\to\infty} P(\tilde{E}_n) = 1 \tag{4.9}$$

where

$$\tilde{E}_n = \{|(Y_{ni}(j) - X_{ni}'(j)\hat{\alpha}_n^{(r+1)} - e_{ni}(j)| \le An^{-1/2+\varepsilon_0},$$
$$j = 1, \ldots, n_i, \quad i = 1, \ldots, c_n\}. \tag{4.10}$$

Now define an estimate of f(0) as follows:

$$f_n(0) = \begin{cases} g_n/(2n^{-1/3}N_n'), & \text{when } E_n \text{ occurs} \\ 0 \quad , & \text{otherwise} \end{cases} \tag{4.11}$$

$$N_n' = n_1 + \ldots + n_{c_n'},$$

and proceed to show that this estimate satisfies (4.4). For this purpose, put

$g_n(a,b)$ = the number of elements in the set

$$\{(i,j):\ a < e_{ni}(j) < b,\ \ j = 1,\ldots,n_i,\ \ i = 1,\ldots,c_n'\}$$

and define

$$f_{n1}(0) = g_n(-n^{-1/3},\ n^{-1/3})/(2n^{-1/3}N_n')$$

$$f_{n2}(0) = g_n(-n^{-1/3} + n^{2\varepsilon_0 - 1/2},\ n^{-1/3} - n^{2\varepsilon_0 - 1/2})/(2n^{-1/3}N_n')$$

$$f_{n3}(0) = g_n(-n^{-1/3} - n^{2\varepsilon_0 - 1/2},\ n^{-1/3} + n^{2\varepsilon_0 - 1/2})/(2n^{-1/3}N_n').$$

From the well-known result in the theory of density estimation (see [8], Chapter 2) and the easy fact that

$$\liminf_{n\to\infty} N_n'/n > 0, \quad \text{a.s.}, \tag{4.12}$$

under the assumption of Section 2.1, we have

$$f_{n1}(0) - f(0) = O_p(n^{-1/3}). \tag{4.13}$$

Since

$$f_{n2}(0) = (1 + O(n^{2\varepsilon_0 - 1/6}))f_{n1}(0)$$

$$f_{n3}(0) = (1 + O(n^{2\varepsilon_0 - 1/6}))f_{n1}(0),$$

from (4.13) we have

$$f_{n2}(0) - f(0) = O_p(n^{2\varepsilon_0 - 1/6})$$

$$f_{n3}(0) - f(0) = O_p(n^{2\varepsilon_0 - 1/6}). \tag{4.14}$$

On the other hand, it is easy to see that when n is large and the event $E_n \cap \tilde{E}_n$ occurs, we have

$$f_{n2}(0) < f_n(0) < f_{n3}(0).$$

Therefore, from (4.7), (4.9) and (4.14), we get

$$f_n(0) - f(0) = 0_p(n^{2\varepsilon_0 - 1/6}). \tag{4.15}$$

But $\sqrt{c_n} = 0_p(n^{1/2-\varepsilon_2/2})$ (see (2.6)), and since $\varepsilon_0 < (\varepsilon_2 - 2/3)/4$, we have $1/6 - 2\varepsilon_0 > 1/2 - \varepsilon_2/2$. From this and (4.15), we finally get (4.4).

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